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On quasi-arithmetic rings†

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Summary

In this short note we characterize the rings (commutative with unity) that satisfy the following property:
(*) for every ideals I and J is $I \cdot J = (I + J) \cdot (I \cap J)$.
In the case of a domain we recover a well known characterization of Prüfer domains. Moreover we give a more explicit description in the case of a noetherian ring.

Key words: *commutative rings, ideals, domains*

Riassunto

In questa breve nota caratterizziamo gli anelli (commutativi e con unità) che godono della seguente proprietà:
(*) per due qualunque ideali I e J si ha $I \cdot J = (I + J) \cdot (I \cap J)$.
Nel caso di un dominio ritroviamo una ben nota caratterizzazione dei domini di Prüfer. Diamo poi una descrizione più esplicita nel caso di un anello noetheriano.

Parole chiave: *anelli commutativi, ideali, domini*

1 Introduction.

In this short note we characterize the rings (commutative with unity) that satisfy the following property:
(*) for every ideals I and J is $I \cdot J = (I + J) \cdot (I \cap J)$.
Those rings are a generalization of the *arithmetic rings* intoduced by J.P. Lafon in [**?**] and will be called *quasi-arithmetic rings*.
In the case of domains we recover a well known characterization of Prüfer domains (see [**?**] Th.25.2).
The paper is organized as follows: in section 2 we give the notions of *arithmetic rings* and *quasi-arithmetic rings*. In section 3 we prove the aforesaid characterization, while in section 4 we give a more explicit description in the case of a noetherian ring.
All the proofs are elementary. All the rings we consider are commutative with unity; all the basic notions and properties about commutative rings can be found in [**?**].

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2 Quasi-arithmetic rings.

In this section we recall the definition of arithmetic ring and extend this to the notion of quasi-arithmetic ring we will use in the next sections.

Definition 1. Let A be a local ring and let M its maximal ideal; we say that A is an *arithmetic ring* if it satisfy one of the equivalent conditions:
i) *the set of ideals of A is linearly ordered by inclusion,*
ii) *the set of principal ideals of A is linearly ordered by inclusion,*
iii) *for all $x, y \in A$, the ideal (x, y) is principal.*
Let us prove the equivalence between i), ii) and iii). Clearly i) implies ii) and ii) implies iii).
Let us prove that iii) implies ii); put $(z) = (x, y)$ and assume $z \neq 0$: we have $z = ax + by, x = cz, y = dz$; if one between c or d is invertible, e.g. c invertible, we have $(z) = (x) \supseteq (y)$; otherwise we have $x, y \in M$ and $z = acz + bdz$, i.e. $(1 - ac - bd)z = 0$; but $1 - ac - bd$ is invertible which implies $z = 0$.
Finally we prove that ii) implies i); let I, J two ideals of A and assume they are not comparable; hence there is $x \in I, x \notin J$ and $y \in J, y \notin I$; since (x) and (y) are comparable we assume $(x) \subseteq (y)$ hence $x \in J$.

Definition 2. Let A be any ring; we say that A is an *arithmetic ring* if A_M is arithmetic, for any maximal ideal M.

Definition 3. Let A be a local ring and let M its maximal ideal; we say that A is a *quasi-arithmetic ring* if it satisfy one of the two equivalent conditions:
i) *for all $x, y \in A$ either $xy = 0$ or the ideal (x, y) is principal,*
ii) *for all $x, y \in A$ either $(x, y)^2 = 0$ or the ideal (x, y) is principal.*
Let us prove the equivalence between i) and ii). Clearly ii) implies i). Let us prove the converse. Assume $(x, y)^2 \neq 0$; if $xy \neq 0$ then the ideal (x, y) is principal; otherwise $xy = 0$ and we may assume $x^2 \neq 0$: in this case we have $x(x + y) \neq 0$ hence the ideal $(x + y, x) = (x, y)$ is principal.

Definition 4. Let A be any ring; we say that A is an *quasi-arithmetic ring* if A_M is quasi-arithmetic, for any maximal ideal M.

3 Characterization of quasi-arithmetic rings.

In this section we prove the characterization of quasi-arithmetic rings given in the introduction.

Proposition 1. Let A be a commutative ring with unity; then the following are equivalent:
i) A is quasi-arithmetic,
ii) A satisfies the following property:
(*) for every ideals I and J is $I \cdot J = (I + J) \cdot (I \cap J)$.

Proof. First we observe that, by elementary commutative algebra, all the operations involved in the equality (*) commute with localization, hence we can assume that A is a local ring.
Now let us prove that i) implies ii). Since it is always true that $I \cdot J \supseteq (I + J) \cdot (I \cap J)$ it is enought to prove the inclusion $I \cdot J \subseteq (I + J) \cdot (I \cap J)$. To this end let $x \in I, y \in J$ and assume $x \cdot y \neq 0$; by i) we have $x = ay$ or $y = bx$; let $x = ay$, then xy is the product of $y \in J \subseteq I + J$ times $x \in I \cap J$.
Now let us prove that ii) implies i). To this end let $x, y \in A$ and assume $(x, y)^2 \neq 0$; we assume also that both x, y are in the maximal ideal M of A; by Nakayama lemma we have:

$$(x, y)^2 M \neq (x, y)^2.$$

The we can assume $xy \notin (x, y)^2 M$: in fact if it is $xy \in (x, y)^2 M$ and $x^2 \notin (x, y)^2 M$ we have $x(x + y) \notin (x, y)^2 M$ and $(x, y) = (x, x + y)$.
We apply (*) to the ideals $(x), (y)$; we have

$$xy \in (x)(y) \subseteq ((x) + (y)) \cdot ((x) \cap (y));$$

assume, by contraddition, that $x \notin (y)$ and $y \notin (x)$ and let z be any element of $(x) \cap (y)$: we have $z = \lambda x = \mu y$ with both $\lambda, \mu \in M$, hence $z \in (x, y)M$ i.e. $(x) \cap (y) \subseteq (x, y)M$; finally we have $((x) + (y)) \cdot ((x) \cap (y)) \subseteq (x, y)^2 M$ and $xy \in (x, y)^2 M$: absurd.

4 The noetherian case.

In this section we give a more explicit description of quasi-arithmetic noetherian local rings.

Proposition 2. Let A be a noetherian local ring and let M be its maximal ideal: then the following are equivalent:
i) A is quasi-arithmetic,
ii) A is one of the following types:
a) a D.V.R. (Discrete Valuation Ring),
b) an artinian ring with M principal,
c) an artinian ring with $M^2 = 0$.

Proof. First prove that ii) implies i). If A is a D.V.R. then A is arithmetic since $M = (x)$ is principal and all non zero ideals are of the form $(x^r), r \geq 0$.
If A is an artinan local ring with $M = (x)$ principal we prove as before that all non zero ideals are of the form $(x^r), r \geq 0$: in fact by Krull intersection theorem is $\cap_{n \geq 0}(x)^n = 0$ so if I is a non zero ideal we put r to be the maximal integer such that $I \subseteq (x)^r$: we have $I = (x)^r$, otherwise for all $y \in I$ we would have $y = \lambda_y x^r$ with $\lambda_y \in M$ hence $\lambda_y = \mu_y x$ and $I \subseteq (x)^{r+1}$.
Finally if A is an artinian ring wwith $M^2 = 0$ then A is quasi-arithmetic: in fact if x, y are two elements in A we have that either x, y are both in M in which case is $xy = 0$ otherwise is $(x, y) = (1)$.
Now let us prove that i) implies ii). If $M^2 = 0$ then A is of the type c) : in fact since M is the only prime and A is noetherian it is also artinian.
Assume from now on $M^2 \neq 0$. We prove that in this case that M is principal. Assume $\{z_1, z_2, \ldots, z_t\}$, $t > 1$ is a minimal system of generators of M; then there exists $1 \leq i, j \leq t$ such that $z_i z_j \neq 0$; we can assume $i \neq j$; in fact if $z_i z_j = 0$, $\forall i \neq j$ and is e.g. $x_1^2 \neq 0$ then $z_1(z_1 + z_2) \neq 0$ and $\{z_1, z_1 + z_2, \ldots, z_t\}$ is also a minimal system of generators of M; A being quasi-arithmetic, $(z_i, z_j) = (w)$ is principal and M is minimally generated by $t - 1$ elements.
By Krull height Theorem two cases are possible. If the height of M is one then A is a regular local ring of dimension 1 hence a D.V.R.. It remain the case when the height of M is zero: in this case M is the only prime of A and A is artinian.

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