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Why are the stars as they are? Eddington's parable revisited.†

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Summary

We revisit the famous Eddington's parable on the existence of the stars published in 1926, well before the nuclear sources of energy in stellar interiors were known. The moral of the parable, namely that the stars can exist only in a limited range of masses, is discussed in the light of Chandrasekhar's theorem enunciated in 1936, that treats the argument in a more rigorous way, but leaving unaltered the Eddington's conclusions. It emerges that the existence of the stars cannot be attributed to the case but to a unique combination of the constants of nature involving special and general relativity, quantum mechanics and atomic physics.

Key words: *Astrophysics, stars, Eddington's parable, Chadrasekhar's theorem*

Riassunto

Si ridiscute la famosa parabola di Eddington sull'esistenza delle stelle pubblicata nel 1926, molto prima che le sorgenti nucleari di energia negli interni stellari fossero conosciute. La morale della parabola, cioé che le stelle possono esistere solo in un limitato intervallo di masse, è discusso alla luce del teorema di Chandrasekhar enunciato nel 1936 che tratta l'argomento in modo più rigoroso pervenendo però alle stesse conclusioni di Eddington. Emerge che l'esistenza delle stelle non può essere attribuita al caso ma ad un'unica combinazione delle costanti della natura relative alla relatività, speciale e generale, alla meccanica quantistica e alla fisica atomica.

Parole chiave: *Astrofisica, stelle, parabola di Eddington, teorema di Chandrasekhar*

1 Introduction

The fundamental parameters of the stars are the mass, M, the radius, R, and the luminosity, L. The latter is the total electromagnetic energy irradiated per unit time into the space by the star and is linked to the so called effective temperature T_e through the Stefan – Boltzmann law, $L = 4\pi R^2 \sigma T_e^4$. T_e is therefore a temperature defined on the basis of the electromagnetic flux flowing through the surface layers of the star and can be determined by its spectrum.

The measurement of the stellar parameters is not an easy task, thought in modern times the astrometric satellite Hipparcos has characterized more than 40 thousand nearby stars up to about

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500 light-years. The mass can be determined directly only for double star systems (visual systems, and in some cases spectroscopic and photometric systems). Since the stars are point-like sources, stellar radii can be measured directly only for a limited amount of nearby objects, say about 50, by interferometry or by utilizing eclipsing binary stars and the rare lunar transits. Otherwise the radius may be estimated from surface gravity derived from the analysis of the line spectrum, once the mass of the star is known. The measure of luminosity implies the detection at ground of the radiation flux produced by the whole electromagnetic stellar emission from the very short wavelengths (X-ray domain) to the longer ones (infrared domain), once the distance of the star is known through the measure of its parallax (Hipparcos determinations). However the Earth's atmosphere prevents the possibility of measuring the integral spectrum, short and long wavelengths being absorbed, permitting the transmission of the only visible light. Some bolometric measurements were carried out from space, essentially for calibrating the bolometric corrections to be applied to the measured flux at ground. For distant stars, for which the parallax method is not applicable, indirect measurements of luminosity are based on the spectral analysis.

As a matter of the fact, the range of the observed stellar fundamental parameters spans several orders of magnitude for the luminosity (~ 12) and radius (~ 5), but only 2 orders of magnitude for the mass. The luminosity ranges from about $10^{-6}\,L_\odot$ for the faintest red dwarfs to about $10^{6}\,L_\odot$ for the most luminous supergiants, and the radius ranges from $10^{-2}\,R_\odot$ for white dwarfs to $10^{3}\,R_\odot$ for red supergiants, where $L_\odot = 3.845 \times 10^{26}$ W and $R_\odot = 6.959 \times 10^{8}$ m are the solar luminosity and radius respectively. The almost totality of masses are observed in the range $0.5\,M_\odot$ – $50\,M_\odot$, where $M_\odot = 1.989 \times 10^{30}$ kg is the solar mass.

Well before the discovery of the neutron in 1932 by Sir James Chadwick, with the consequent deeper knowledge of the atomic nucleus, and the establishment of thermonuclear reactions as energy source in stellar cores in 1938 by Hans Bethe, in 1926 was published the book *The internal constitution of the stars* by Sir Arthur Eddington [1]. At that time the only energy source of the stars was believed to be the gravitational one so that the life of a stars was determined by Kelvin – Helmoltz time $\tau_{\rm KH} \simeq GM^2/RL$, where G is the gravitation constant. This would imply a life of about 30 million years for the Sun and a life inversely proportional to the mass for other stars, being $L \propto M^3$ from homology relationships [1]. Now we know that the life of the stars is marked by its nuclear time $\tau_{\rm n} \simeq qfM/L$, where q is the energy released per unit mass by nuclear fuel and f is the fraction of mass in which the fuel is burnt. This would imply a nuclear life of about 10 billion years for the Sun and a life inversely proportional to the square of the mass. Though he was completely unaware of the nuclear processes, Eddington had the intuition that, beside the gravitational energy, some other process involving atomic nuclei could be at work to sustain the stellar luminosity for times much longer than those predicted by the slow gravitational energy release, also because it was already known from geological studies that the age of the Sun was of the order of billion and not million years.

At Eddington's times stellar models, namely the spherical distribution of mass, density and pressure from the centre to the surface, were constructed for different polytropic structures [2] by integrating the Lane – Emden equation [2] that solves for the gravitational potential and is based on mass continuity and hydrostatic equilibrium equations, disregarding energy production and transport, essential ingredients of modern stellar models.

Eddington noted that the large dispersion of luminosity and radius observed values was in striking contrast with the narrow range of values determined for stellar masses, and thought that a basic physical reason should prevent the existence of the stars outside this range, that is the answer to the question "why are the stars as they are?".

This argument has been discussed more recently also by Srinivasan [3] in the context of the scientific work of the 1983 Nobel prize Subrahmanyan Chandrasekhar and by Paternò [4] in the

[1] Homologous stars are those that have the same distribution of mass with respect their radius; this implies $\varrho \propto M/R^3$, $P \propto M^2/R^4$, $T \propto M/R$, where ϱ is the density, P the pressure, and T the temperature.

[2] A polytropic structure is characterized by an assigned functional relationship between pressure, P, and density, ϱ, through a polytropic index n, such that $P = K\varrho^{\gamma}$, with K a constant and $\gamma = (n+1)/n$.

context of the antropic principle.

2 Eddington's parable

Eddington's parable [1] describes a physicist who lives on a planet permanently covered by clouds so that for him it is impossible to see the sky and the stars. Notwithstanding he is able to discover how the stars are structured, even if he never heard of them. Eddington imagines that the physicist can calculate the gas, P_g, and the radiation, P_r, pressures of a series of gas globes, starting from a globe of 10 g and considering successively masses of 10^2 g, 10^3 g in such a way that the n$^{\text{th}}$ globe has the mass of 10^{n} g.

Our physicist constructs a table in which he reports in two columns the values of P_r/P_t (2$^{\text{nd}}$ column) and P_g/P_t (3$^{\text{rd}}$ column) for each globe n (1$^{\text{st}}$ column), where $P_t = P_r + P_g$ is the total pressure. The results of original Eddington's calculations are shown in Table 1; the table appears to be monotonous until the globe No. 32 with the radiation pressure numbers all close to zero and the gas pressure numbers all close to one and it restarts to be monotonous from globe No. 36 on, but with inverted columns, namely with radiation pressure numbers close to one and gas pressure numbers close to zero. Eddington observes that something of strange with respect to the remaining parts of the table happens for the globes Nos. 33, 34, and 35, where there is competition between radiation and gas pressure, and concludes that what is strange are the stars.

Table 1: Original Eddington's table for radiation and gas pressures as functions of the gas globe number.

No. of globe	Radiation pressure	Gas pressure
30	0.00000016	0.99999964
31	0.000016	0.999984
32	0.0016	0.9964
33	0.106	0.894
34	0.570	0.430
35	0.850	0.150
36	0.951	0.049
37	0.984	0.016
38	0.9951	0.0049
39	0.9984	0.0016
40	0.99951	0.00049

If now the cloud veil is dissolved, our physicist will see a multitude of luminous globes of gas nearly all of masses between his No. 33 and No. 35 globes, namely between 0.5 $M_\odot$ and 50 $M_\odot$.

In his book [1] Eddington describes how the physicist calculated the data reported in Table 1. The gas pressure can be expressed by the following relationship:

$$P_g = \frac{k\varrho T}{\mu m_u} \tag{1}$$

where k is the Boltzmann constant, ϱ the density, T the temperature, μ the molecular weight, and m_u the atomic mass unit. The radiation pressure can be expressed by the following expression:

$$P_r = \frac{1}{3}aT^4 \tag{2}$$

where a is the radiation density constant:

$$a = \frac{8\pi^5 k^4}{15c^3h^3} \tag{3}$$

with c the vacuum light speed and h the Planck constant. By defining $P_g = \beta P_t$ and therefore $P_r = (1-\beta)P_t$, with $0 \leq \beta \leq 1$, and eliminating T from Eqs. (1) and (2) one obtains:

$$P_t = \left[\frac{3}{a}\,\frac{1-\beta}{\beta^4}\left(\frac{k}{\mu m_u}\right)^4\right]^{\frac{1}{3}} \varrho^{\frac{4}{3}} \tag{4}$$

For any fixed value of β, Eq.(4) has the same form of a polytropic relationship with $\gamma = 4/3$ and $n = 3$, and K the constant factor multiplying $\varrho^{4/3}$ in Eq.(4) (see footnote 2 in Section 1). The theory of polytropic structures indicates that K is linked to the central gravitational potential of the star, ϕ_0, through the following relationship:

$$K = \frac{1}{4}\left(4\pi G\Lambda^2\phi_0^2\right)^{\frac{1}{3}} \tag{5}$$

where Λ is a scaling parameter for lengths and $\phi_0 = GM/\Lambda M'$, with $M' = 2.0182$ obtained by the numerical integration of Lane – Emden equation for $n = 3$. By comparing K of Eq.(4) with K of Eq.(5), Eddington obtained an equation that relates the mass of the star to the mixture of radiation and gas pressures β for any mean molecular weight μ:

$$M = \frac{4M'}{\pi^{1/2}G^{3/2}}\left[\frac{3}{a}\,\frac{1-\beta}{\beta^4}\left(\frac{k}{\mu m_u}\right)^4\right]^{\frac{1}{2}} \tag{6}$$

By inserting the values of the constants that appear in Eq.(6) one obtains the equation for calculating Table 1:

$$M_{34} = \frac{3.6379}{\mu^2}\left(\frac{1-\beta}{\beta^4}\right)^{\frac{1}{2}} \tag{7}$$

where M_{34} is the mass of the star expressed in terms of 10^{34} g, namely the weight of the globe No. 34. For constructing his table through Eq.(7), Eddington used $\mu = 4$ that, in the light of the modern knowlegdes, is an unrealistic molecular weight for the interior layers of the stars. At Eddington's times it was belived that interiors of the stars were mainly constituted by heavy elements like carbon, oxigen and iron, while today we know that the main ingredients are hydrogen and helium ionized with a tiny quantity of heavy elements and $\mu \simeq 1$.

What the parable teaches us is that the stars can exist only in a restricted interval of masses in which there is the proper balance between radiation and gas pressure; if the radiation dominates the star cannot stay in mechanical equilibrium and is destroyed by its radiation wind, while if radiation is negligible the star is too cool for burning its hydrogen fuel and collapses.

3 Chandrasekhar's theorem

The thought of Eddington appears clear in the light of more rigorous considerations. It is well known that the stars, during their stability phase, are in mechanical equilibrium since the pressure generated by the weight of the layers of matter, the gravitational pressure, is exactly balanced by the sum of radiation and gas pressures. The gravitational pressure, namely the pressure exerted at the centre of the star by the overlying layers, P_c, is obtained by integrating the hydrostatic equilibrium equation:

$$P_c = \frac{G}{4\pi}\int_0^M \frac{M(r)dM(r)}{r^4} \tag{8}$$

where $M(r)$ is the mass contained in a sphere of radius r. From Eq.(8) Chandrasekhar [5] in 1936 derived a general theorem on the limits of central pressure in stars. He demonstrated that for a star in mechanical equilibrium, for which the density does not increase from the centre to the surface, $\varrho(r) \leq \overline{\varrho}(r)$, the central pressure must lay between a lower limit corresponding to a structure with a constant density equal to the mean density of the star, $\bar{\varrho}(R)$, and a higher limit corresponding to a structure with a constant density equal to the central density of the star ϱ_c:

$$\frac{1}{2}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} GM^{\frac{2}{3}}\,[\bar{\varrho}(R)]^{\frac{4}{3}} \leq P_c \leq \frac{1}{2}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} GM^{\frac{2}{3}}\varrho_c{}^{\frac{4}{3}} \tag{9}$$

Since in equilibrium $P_c = P_t$, it is possible to substitute P_c of Eq.(9) with P_t defined in Eq.(4) obtaining:

$$\frac{1}{2}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} GM^{\frac{2}{3}}\,[\bar{\varrho}(R)]^{\frac{4}{3}} \leq \left[\frac{3}{a}\,\frac{1-\beta}{\beta^4}\left(\frac{k}{\mu m_u}\right)^4\right]^{\frac{1}{3}} \varrho^{\frac{4}{3}} \leq \frac{1}{2}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} GM^{\frac{2}{3}}\varrho_c{}^{\frac{4}{3}} \tag{10}$$

Inequality (10), derived from Chandrasekhar's theorem, shows that cannot exist stars with the only radiation pressure ($\beta = 0$) or gas pressure ($\beta = 1$) supporting the gravitational pressure; in the first case inequality (10) is satisfied for $M = \infty$ while in the second case for $M = 0$, both being unphysical cases. If we consider the upper limit of the inequality (10) by replacing ϱ with ϱ_c, we obtain a relationship that links the mass of the star to the proper value of β compatible with that mass:

$$M^\star = \frac{\sqrt{135}}{2\pi^3}\left(\frac{ch}{Gm_u^{4/3}}\right)^{\frac{3}{2}} \frac{1}{\mu^2}\left(\frac{1-\beta}{\beta^4}\right)^{\frac{1}{2}} \tag{11}$$

where $M^\star$ is the mass expressed in terms of solar mass and the radiation density constant a has been expressed trough Eq.(3). By inserting the values of constants appearing in Eq.(11) we finally obtain:

$$M^\star = \frac{5.549}{\mu^2}\left(\frac{1-\beta}{\beta^4}\right)^{\frac{1}{2}} \tag{12}$$

that is the same expression as that found by Eddington by using a different procedure based on the polytropic similarity instead of a general theorem, that incidentally did not exist when Eddington published his work, as in this case. The coefficients of Eq.(7) and Eq.(12) differ because a different mass scaling has been adopted; in the first case a mass of 10^{34} g corresponding to the Eddington's globe No. 34 while in the second case the solar mass, $M_\odot = 1.989\times10^{30}$ kg. However, if we adopt the same scaling as Eddington's procedure in the procedure based on Chandrasekhar's theorem, we still find a difference in the coefficients, Eddington's coefficient of Eq.(7) being larger by a factor of about 3.3, that points out the difference of the two methods, the second being more rigorous, but does not change the conclusions.

We solved Eq.(12) numerically by Newton – Raphson method for $\mu = 1$ and obtained the results shown in Table 2 where it is clear that in the range of observed masses there is a competition between radiation and gas pressure that insures the stellar equilibrium. The numbers are different from those reported in the original Eddington's table that depends on a different choice of μ, but the conclusion that is necessary a proper mixture of pressures for maintaining equilibrium is confirmed.

4 Luminosity and mass limits

The problem of the mass limits, that involves the problem of the luminosity limit, can also be discussed by making use of the following argument that may better clarify the scenario. In the simplest case, as that examined by Eddington, we can imagine a pure hydrogen star not very different from the real case of hydrogen (70%) and helium (28%) star with a tiny quantity (2%) of heavier elements. Owing to the large temperature and pressure in the interior, hydrogen is fully ionized in a large part of the stellar mass with the lighter electrons being affected by radiation pressure and heavier protons affected by gravitational force that tends to separate these particles,

Table 2: Modern table for radiation ($1-\beta$) and gas (β) pressures as functions of the stellar masses in terms of the solar mass $M/M_\odot$. In between the horizontal lines there are the masses corresponding to the globes 33 – 35 of Eddington's Table 1.

$M/M_\odot$	$1-\beta$	β
0.01	0.000003	0.999997
0.02	0.000013	0.999987
0.05	0.000081	0.999919
0.1	0.000324	0.999676
0.2	0.001292	0.998708
0.5	0.007867	0.992133
1	0.028885	0.971115
2	0.089344	0.910656
5	0.252923	0.747077
10	0.405551	0.594449
20	0.547011	0.452989
50	0.695749	0.304251
100	0.778717	0.221283
200	0.840512	0.159488

the electrons being expelled from the star and the protons being attracted toward the centre of the star, that leads to the breaking of the star.

The force exerted by the radiation flux $S = L/4\pi r^2$ on the electrons is $F_r = S\sigma_T/c$, where σ_T is the Thomson cross section for electron scattering, while the gravitation force acting on the protons is $F_g = Gm_pM/r^2$, where m_p is the proton mass; the luminosity limit, called Eddington luminosity, L_E, is obtained by equating the two forces:

$$L_E = \frac{4\pi c G m_p M}{\sigma_T} \cong 3.27\times 10^4 \left(\frac{M}{M_\odot}\right) L_\odot \tag{13}$$

From Eq.(13) by using the homology relation $L \propto M^3$ it is possible to determine the maximum mass compatible with the radiation pressure, that gives $M \simeq 180\, M_\odot$, with $1-\beta < 0.84$ as from Table 2. The lower mass limit, M_{min} is linked to the minimum temperature, T_{min} for hydrogen burning that recently has been estimated to be about $T_{min} = 4\times 10^6\, K$ [6]. In the long living, stable, main sequence phase the stars are nearly homologous with a nearly constant surface gravity $g = GM/R^2$ that implies $M \propto R^2$, and $M \propto T^2$. Therefore, by using the present solar value for central Sun's temperature, $T_{c\odot} = 1.57\times 10^7\, K$ [7], we obtain:

$$M_{min} = \left(\frac{T_{min}}{T_{c\odot}}\right)^2 M_\odot \simeq 0.065\, M_\odot \tag{14}$$

Therefore it is possible the existence of stars between the limits $0.065\, M_\odot$, as *Gliese 570 D, V1581 Cygni C, Denis 1048 –39,* and $180\, M_\odot$, as *Eta Carinae, Pistol star, R136 C,* but the large majority of stars have masses in the range predicted by Eddington and confirmed by Chandrasekhar theorem.

5 Conclusion

Equation (11) shows that the existence of the stars is not due to the case but to a unique combination of the constants of nature G (constant of Newtonian mechanics and general relativity), c (constant of special relativity), h (constant of quantum mechanics), and m_u (constant of atomic physics). The combination of these constants, $(ch/Gm_u^{4/3})^{3/2}$, gives a mass of about $30\, M_\odot$ for which there is equilibrium between radiation and gas pressures and corresponds to a mass in between the Eddington's globes Nos. 34 and 35.

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